Network Voronoi Graph and Closest Point Problem

Maaza Mekuria
Northeastern University
What is the Problem?

• Stop Spacing Optimization is not a simple one dimensional model
• Needs more study of curve effect
Problem Definition

Parcel i

Destination

Stop j

Stop k

Origin

\[ W_{ik} \]

\[ W_{ij} \]

\[ R_k \]

\[ R_j \]
Network Definitions

- Overview of properties of a Voronoi diagram in the Euclidean space \([1]\)
- The set of generator points are more than 2 but finite i.e. \(2 \leq n < \infty\)
- Voronoi diagram (sometimes known as Thiessen Polygons) shows the closest region to a point in the set
- This is expressed mathematically as, if \(p_1, \ldots, p_n\) are points in the set with location vectors \(x_1, \ldots, x_n\) and \(x_i \neq x_j\), where \(i \neq j\), \(i, j \in \{1, \ldots, n\}\) and \(p\) be any point in the plane then \(p\) is exclusively assigned to \(p_i\) if and only if \(||x-x_i|| < ||x-x_j||\) where
Network Definitions

- The boundary is formed by the point set (or edge) that is equidistant from two or more members. i.e. $||x-x_i|| = ||x-x_j||$
- The assigned region is collectively exhaustive since every space in the region will be assigned, and it is exclusive and is called a tessellation.
- $V(p_i)=\{x||x-x_i|| \leq ||x-x_j|| \text{ for } i \neq j, j \in \text{In}\}$
- The Tessellation is called an ordinary planar Voronoi diagram, mathematically
Measures on Euclidean Space

• Some measures of Voronoi diagrams are
• in the Euclidean metric the distance between \( p \) and \( p_i \) is given by
  \[ d_e(p,p_i) = \| x - x_i \| = \sqrt{(x_1 - x_{11})^2 + (x_2 - x_{12})^2} \]
• in the Manhattan metric
  \[ d_m(p,p_i) = \| x - x_i \| = \sum_{j=1}^{m} |x - x_j| \]
More measures

• The Karlsruhe metric if polar coordinates $p, p_i$ ($0 \leq \theta, \theta_i < 2\pi$, $r, r_i > 0$ and $\delta(\theta, \theta_i) = \min(|\theta - \theta_i|, 2\pi - |\theta - \theta_i|)$, then

\[
d_k(p, p_i) = \min \{r, r_i\} \delta(\theta, \theta_i) + |r-r_i| \quad \text{for} \quad 0 \leq \delta(\theta, \theta_i) < 2
\]

• $r + r_i \quad \text{for} \quad 2 \leq \delta(\theta, \theta_i) < \pi$
Network Measure

- Voronoi diagram on a network –
- Let $G(N,L)$ be a planar geometric graph with as set of nodes,
- $N=\{p_1,p_2,\ldots,p_n,p_{n+1},\ldots,p_l\}$, where $P=\{p_1,p_2,\ldots,p_n\}$ are the stations (generators)
Network Voronoi Link diagram

– Dominance,
\[ \text{Dom}(p_i, p_j) = \left\{ p \mid p \in \bigcup_{i=1}^{k} l_i, d_{\text{net}}(p, p_i) \leq d_{\text{net}}(p, p_j), j \in I_n \setminus \{i\} \right\} \]

• Boundary,
\[ b(p_i, p_j) = \left\{ p \mid p \in \bigcup_{i=1}^{k} l_i, d_{\text{net}}(p, p_i) = d_{\text{net}}(p, p_j), j \in I_n \setminus \{i\} \right\}, i \neq j \]

• Link Voronoi diagram,

• \( \text{Vlink}(p_i) = \bigcap_{j \in I_n \setminus \{i\}} \text{Dom}(p_i, p_j) \)
Case Study: MBTA Green Line - "B" - Comm Ave Parcel Boundary for Inbound Boarding at Fordham and Harvard (PM period)
Case Study: MBTA Green Line - "B" - Comm Ave Parcel Boundary for Inbound Alightings at Fordham and Harvard (PM period)
Network Voronoi Area Diagram

- The network Voronoi link diagram

- Access distance from point p,
  \[ d_{\text{acc}}(p,a(p)) = \| x - x_a \| = \min_u \{ \| x - u \| : u \bigcup_{i=1}^k l_i \} \]

- Network distance
  \[ d_{\text{acc.net}}(p,p_i) = \left\{ p \in \bigcup_{i=1}^k l_i, d_{\text{net}}(p, p_i) = d_{\text{net}}(p, p_j), j \in I_n \setminus \{i\}, i \neq j \right\} \]

- Voronoi area
  \[ V_{\text{area}}(p_i) = \left\{ p \mid d_{\text{acc.net}}(p, p_i) \leq d_{\text{acc.net}}(p, p_j), j \in I_n \setminus \{i\} \right\} \]
Algorithmic Solutions

• Algorithmic solution to the network Voronoi graph
• Weighted Voronoi Diagram for Transportation described on [1] modified
Continous Dijkstra

• \( d_T(p,p_i) = \min\{d_{acc}(p,p_j)+d_T(p_i,p_j)+r(p_i)\} \), for \( j=1,...,l \)
  \[ i \in |P| = I_n \]

\( r(p_i) \) is the cost of the travel on the transit network until the end of the line in the desired direction.
Parallel Dijkstra

• An algorithm is proposed by Boratyn, Melachrinoudis, and Min using a parallel Dijkstra with a Fibonacci heap based implementation.
Geographic Implementation
Euclidean Distance Voronoi Map for the Station Set in Boston.
Minimum Total Travel Time Graph using Network Metrics
Comparison of Simple Shortest Path (SSP) and Network Voronoi (NV)

• NV has less computation requirement initially
• NV enables easier addition and processing of data
• NV has over head for determination of access distance locally later on.
Advantages of Network Voronoi

- Richer set of data with less computation time
- Able to provide visual interpretation for the information
- Models various scenarios such as mix-mode service etc with ease
References

• [1] Spatial Tessalations – Okabe, Boots, Sugihara and Chiu, 2000